Related Rates

EXAMPLE 1 Finding Related Rate Equations

(a) Assume that the radius r of a sphere is a differentiable function of t and let V be the volume of the sphere. Find an equation that relates dV/dt and dr/dt.

(b) Assume that the radius r and height h of a cone are differentiable functions of t and let V be the volume of the cone. Find an equation that relates dV/dt, dr/dt, and dh/dt.

a) model : $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$$

b) model : $V = \frac{1}{3}\pi \underbrace{r^2 h}$

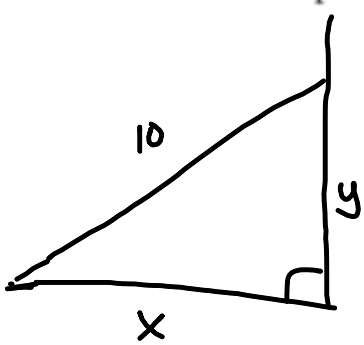
$$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + h(2r)\frac{dr}{dt}\right)$$

EXPLORATION 1 **The Sliding Ladder**

A 10-foot ladder leans against a vertical wall. The base of the ladder is pulled away from the wall at a constant rate of 2 ft/sec.

~~What minimum and maximum values of T make sense in this problem?~~

Find dy/dt when $t = 3$ and interpret its meaning. Why is it negative?



$\frac{dx}{dt}$: rate of change of ground distance from the wall
2ft/sec

$\frac{dy}{dt}$: rate of change of the distance of the ladder from the ground

distance is decreasing so $\frac{dy}{dt}$ is always −

Strategy for Solving Related Rate Problems

1. **Understand the problem.** In particular, identify the variable whose rate of change you *seek* and the variable (or variables) whose rate of change you *know*.

2. **Develop a mathematical model of the problem.** Draw a picture (many of these problems involve geometric figures) and label the parts that are important to the problem. *Be sure to distinguish constant quantities from variables that change over time.* Only constant quantities can be assigned numerical values at the start.

3. **Write an equation relating the variable whose rate of change you seek with the variable(s) whose rate of change you know.** The formula is often geometric, but it could come from a scientific application.

4. **Differentiate both sides of the equation implicitly with respect to time t.** Be sure to follow all the differentiation rules. The Chain Rule will be especially critical, as you will be differentiating with respect to the parameter t.

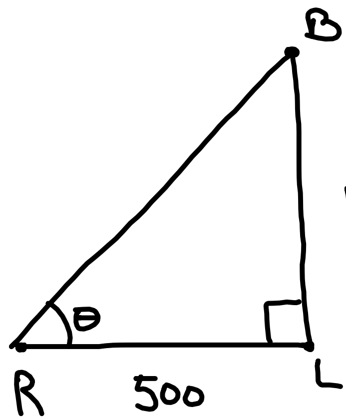
5. **Substitute values for any quantities that depend on time.** Notice that it is only safe to do this *after* the differentiation step. Substituting too soon "freezes the picture" and makes changeable variables behave like constants, with zero derivatives.

6. **Interpret the solution.** Translate your mathematical result into the problem setting (with appropriate units) and decide whether the result makes sense.

EXAMPLE 2 A Rising Balloon

A hot-air balloon rising straight up from a level field is tracked by a range finder 500 feet from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 radians per minute. How fast is the balloon rising at that moment?

plug this value in after you differentiate.



$$\frac{d\theta}{dt} = .14 \text{ radians/min}$$

$$\frac{dh}{dt} = ? \text{ ft/min}$$

$$\tan\theta = \frac{h}{500}$$

$$\tan\theta = \frac{1}{500} h$$

$$h = 500 \tan\theta$$

$$\frac{dh}{dt} = 500 \sec^2\theta \frac{d\theta}{dt}$$

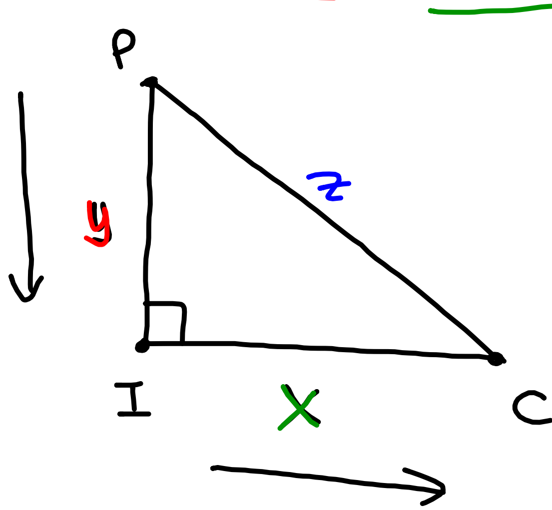
$$= 500 \sec^2\left(\frac{\pi}{4}\right)(.14)$$

$$= 500(2)(.14) = 140$$

$$\frac{dh}{dt} = 140 \text{ ft/min}$$

EXAMPLE 3 A Highway Chase

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?



$\frac{dy}{dt} = -60$ mph

$\frac{dx}{dt} = ?$

$\frac{dz}{dt} = 20$mph

at the instant

$x = .8$ mi

$y = .6$ mi

$z = \sqrt{(.8)^2 + (.6)^2}$

$$x^2 + y^2 = z^2$$

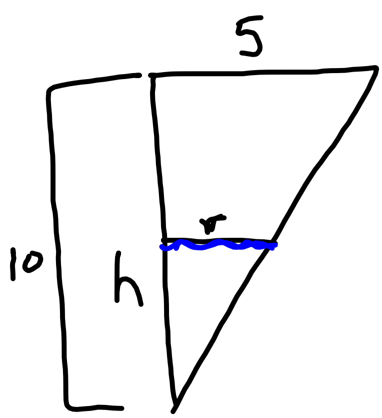
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$2(.8)\left(\frac{dx}{dt}\right) + 2(.6)(-60) = 2(1)(20)$$

$$\frac{dx}{dt} = \boxed{70 \text{ mph}}$$

EXAMPLE 4 Filling a Conical Tank

Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{5}{10} = \frac{r}{h}$$

$$5h = 10r$$

$$h = 2r$$